Engineering Notes

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A Vectorial Form for the Conic Variational Equations

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Introduction

VARIATIONAL equations describing deviations from a reference orbit play a central role in space flight analysis. They are used heavily for error estimation in targeting problems,[1] and they have many applications to optimal orbital transfer and spacecraft rendezvous problems.[2]

Many different derivations and formulations of the variational equations for conic orbits have been given in the literature (e.g., Refs. 2-4). There still seems to be room for improvement, however. Since the variational equations have so many uses, it is important to have them expressed in a compact form where all terms have a clear physical interpretation to facilitate deployment of the equations.

This Note derives a new vectorial form for the conic variational equations in which each term has a definite geometrical interpretation. The result is related to Marec's vectorial form in the Appendix as well as to the more widely used state transition matrix.

Orbital Elements

We will be concerned here with orbits generated by the equation of motion

$$\dot{\boldsymbol{v}} = -\mu(\boldsymbol{r}/r^3) \tag{1}$$

where $\boldsymbol{v}=\dot{\boldsymbol{r}}$, $r=|\boldsymbol{r}|$, and μ is a positive scalar constant. Each orbit is completely characterized by two vector constants of motion, an angular momentum vector

$$\boldsymbol{h}=\boldsymbol{r}\times\boldsymbol{v} \tag{2}$$

and an eccentricity vector

$$\boldsymbol{e}=\mu^{-1}\boldsymbol{v}\times\boldsymbol{h}-\hat{\boldsymbol{r}} \tag{3}$$

where $\hat{\boldsymbol{r}}=r^{-1}\boldsymbol{r}$. The energy (per unit mass)

$$E=\tfrac{1}{2}v^2-(\mu/r) \tag{4}$$

is also a constant of motion, but it is related to $\boldsymbol{h}$ and $\boldsymbol{e}$ by

$$a=\frac{h^2}{\mu(1-e^2)}=-\frac{\mu}{2E} \tag{5}$$

This equation expresses the energy in terms of the geometrical parameter a describing the *size* (and type) of the orbit. The orbit's *shape* is described by the eccentricity $e=|\boldsymbol{e}|$, while its *attitude* or orientation in space is described by the unit vectors $\hat{\boldsymbol{e}}$ and $\hat{\boldsymbol{h}}$. Thus the orbital elements a, e, $\hat{\boldsymbol{h}}$, and $\hat{\boldsymbol{e}}$ provide a complete and direct geometrical characterization of a given orbit.

Variations of the Elements

Variations of the angular momentum $\boldsymbol{h}$ and the eccentricity vector $\boldsymbol{e}$ are not independent of one another because of the constraint $\boldsymbol{h}\cdot\boldsymbol{e}=0$. The constraint can be satisfied by writing the variations in the form

$$\delta\boldsymbol{h}=\delta\boldsymbol{\theta}\times\boldsymbol{h}+\hat{\boldsymbol{h}}\delta h \qquad \delta\boldsymbol{e}=\delta\boldsymbol{\theta}\times\boldsymbol{e}+\hat{\boldsymbol{e}}\delta e \tag{6}$$

These equations can be solved for the angular variation $\delta\boldsymbol{\theta}$. The first equation yields

$$\boldsymbol{h}\times\delta\boldsymbol{h}=h^2\delta\boldsymbol{\theta}-\boldsymbol{h}\boldsymbol{h}\cdot\delta\boldsymbol{\theta}$$

The second equation yields

$$\boldsymbol{h}\times\delta\boldsymbol{e}=\boldsymbol{h}\times\hat{\boldsymbol{e}}\delta e-\boldsymbol{e}\boldsymbol{h}\cdot\delta\boldsymbol{\theta}$$

from which we obtain

$$(\boldsymbol{h}\times\boldsymbol{e})\times(\boldsymbol{h}\times\delta\boldsymbol{e})=\boldsymbol{h}(\boldsymbol{h}\times\boldsymbol{e})\cdot\delta\boldsymbol{e}=\boldsymbol{h}e^2\boldsymbol{h}\cdot\delta\boldsymbol{e}$$

whence,

$$\delta\boldsymbol{\theta}=\frac{\boldsymbol{h}\times\delta\boldsymbol{h}}{h^2}+\boldsymbol{h}\frac{(\boldsymbol{h}\times\boldsymbol{e})\cdot\delta\boldsymbol{e}}{h^2e^2} \tag{7}$$

This equation is of great value in perturbation theory, where $\delta\boldsymbol{\theta}$ can be expressed in terms of the perturbing forces. Here it simply gives us the angular variation of an orbit in terms of the vectorial variations $\delta\boldsymbol{h}$ and $\delta\boldsymbol{e}$. The first term on the right side of Eq. (7) specifies a tilt of the orbital plane, while the second term specifies a rotation of the orbit within the orbital plane.

The variation δa in the size of an orbit is determined in terms of δh and δe by taking the variation of Eq. (5), whence

$$\frac{\delta a}{a}=\frac{2}{h^2}(\boldsymbol{h}\cdot\delta\boldsymbol{h}+\mu a\boldsymbol{e}\cdot\delta\boldsymbol{e})=\frac{2}{h}\delta h+2\frac{\mu ae}{h^2}\delta e \tag{8}$$

Now we can express any orbital variation in terms of the *independent variations* δa, δe, and $\delta\boldsymbol{\theta}$ determining *size, shape,* and *attitude* change.

Variation of Position

The position $\boldsymbol{r}$ of a body on an *elliptical orbit* can be expressed in terms of the orbital elements by the parametric equation

$$\boldsymbol{r}=a[\hat{\boldsymbol{e}}(\cos\phi-e)+\sqrt{(1-e^2)}\,\hat{\boldsymbol{h}}\times\hat{\boldsymbol{e}}\sin\phi] \tag{9}$$

The eccentric anomaly ϕ is related to the time t or the mean anomaly $M=n(t-\tau)$ by Kepler's equation

$$\phi-e\sin\phi=\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}}(t-\tau)=M \tag{10}$$

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Taking the variations of Eqs. (9) and (10), we obtain the variation of position δr in the form

$$\delta r = \frac{\delta a}{a} r - a\delta e\left\{\hat{e} + \frac{\mu e}{h^2} r\cdot\hat{b}\hat{b}\right\} + \delta\theta\times r + v\frac{\delta M}{n} \tag{11}$$

where $\hat{b} = \hat{h}\times\hat{e}$ and

$$\frac{\delta M}{n} = \frac{\partial t}{\partial\phi}\delta\phi = \frac{r}{n\alpha}\delta\phi = \frac{a}{h} r\cdot\hat{b}\delta e - \frac{3}{2}(t-\tau)\frac{\delta a}{a} + \delta t \tag{12}$$

Equation (11) is the central result of this Note. It expresses δr explicitly in terms of variations in orbital size, shape, attitude, and location. Obviously, the first of the four terms on the right side of Eq. (11) describes a *variation of size,* while the third term describes a *variation of attitude.* The second term describes a *variation of shape.* It has two parts: the part proportional to $\hat{e}$ describes a shift in distance between the center and the focus of the ellipse while a is held constant; the other part describes a thinning or fattening of the ellipse associated with this shift. The fourth term describes a shift in the location of a body on a given orbit. The location can be specified by M, ϕ, or t, whichever is most convenient in the problem at hand. Accordingly, the variation in location is given by δM, $\delta\phi$, or δt. In many applications one is interested in comparing locations at the same time only. In this case $\delta t = 0$ and δM is not an independent variation as Eq. (12) shows.

For a body on a hyperbolic orbit, Eqs. (9) and (10) must be replaced by

$$r = a[\hat{e}(e - \cosh\phi) + \sqrt{(e^2-1)}\,\hat{b}\sinh\phi] \tag{13}$$

and

$$e\sinh\phi - \phi = \left|\frac{\mu}{a^s}\right|^{1/2}(t-\tau) = M \tag{14}$$

Variations of these equations again yield Eqs. (11) and (12). Thus, our variational equations (11) and (12) hold for hyperbolic as well as elliptic equations. It would be desirable to generalize the result to variations relating hyperbolic to elliptic orbits, but that will not be attempted here.

The variation in velocity can be obtained directly by differentiating Eqs. (11), with the result

$$\delta v = \frac{\partial}{\partial t}(\delta r) = \frac{\delta a}{a} v - \delta e\frac{\mu a e}{h^2} v\cdot\hat{b}\hat{b} + \delta\theta\times v - \frac{\mu r}{r^3}\frac{\delta M}{n} \tag{15}$$

As before, the four terms on the right side of this equation describe variations in size, shape, attitude, and location.

Both δr and δv can be expressed as functions of r or $\hat{r}$ by using

$$v = (\mu/h^2)h\times(e+\hat{r}) \tag{16}$$

to eliminate v from Eqs. (11) and (15), and

$$r = (h^2/\mu)(1 + e\cdot\hat{r}) \tag{17}$$

to eliminate r.

Propagation of Variations

In targeting or error propagation problems, the state variations δr and δv at a time t are to be determined from the variations δr_0, δv_0 of some initial state r_0, v_0. The desired result is most easily obtained from Eqs. (11) and (15) by first computing variations of the elements from the initial state variations using

$$\delta h = \delta r_0\times V_0 + r_0\times\delta v_0 \tag{18}$$

and

$$\delta e = \mu^{-1}(\delta v_0\times h + v_0\times\delta h) - \delta\hat{r}_0 \tag{19}$$

These equations are simply variations of Eqs. (2) and (3). Of course, h and e can be computed from r_0 and v_0 by using Eqs. (2) and (3) directly. Then Eqs. (18) and (19) can be used in Eqs. (7) and (8) to compute $\delta\theta$ and δa in addition to $\delta e = \hat{e}\cdot\delta e$. With these results inserted in Eqs. (11) and (15) as well as in Eq. (12) with $\delta t = 0$, to get definite values for δr and δv we need only to evaluate v and r from Eqs. (16) and (17) by determining $\hat{r} = \hat{r}(t)$. For a specified time t, Kepler's equation can be used to compute the true anomaly $\theta = \theta(t)$, then $\hat{r} = \hat{r}(t)$ is given by

$$\hat{r} = \hat{e}\cos\theta + \hat{b}\sin\theta \tag{20}$$

Since $(t-\tau)$ is the time measured from periapsis, we must also use Kepler's equation to compute the initial time t_0 from $\cos\theta_0 = \hat{e}\cdot\hat{r}_0$.

We have specified a two-step procedure for computing δr and δv at a specified time t from given initial variations δr_0 and δv_0. First, the element variations δa, δe, $\delta\theta$, and δM are computed from δr_0 and δv_0. Then δr and δv are computed from the element variations. Of course, the element variations can be eliminated by direct substitution to get δr and δv as explicit functions of δr_0 and δv_0, that is, equations of the form

$$\delta r = T_1(\delta r_0) + T_2(\delta v_0) \tag{21}$$

$$\delta v = T_3(\delta r_0) + T_4(\delta v_0) \tag{22}$$

where, for $\lambda = 1,2,3,4$, the T_λ are vector-valued linear functions. Explicit expressions for the T_λ are easy to get, but they are unwieldy and uninstructive, so we will not bother to generate them. It is better to avoid them when possible and to use the indirect method of relating state variations to variations of the elements.

From our vectorial equations the derivation of the usual state transition matrix for any coordinate system is straightforward. Let $[e_k$, for $k = 1,2,3]$ be an orthonormal frame for some coordinate system. The coordinate system may be fixed as is usual or rotating as in Jones' formulation.[4] The transition matrix for this system is given by

$$e_j\cdot T_\lambda(e_k) \tag{23}$$

For $\lambda = 1,2,3,4$, this is a set of four 3×3 blocks of the 6×6 transition matrix. Of course, there is no need to compute the transition matrix if the vectorial approach described above is used.

Appendix: The Primer Vector

The primer vector p_V plays a central role in orbit optimization theory as developed, for example, by Marec.[2] The primer vector is proportional to an orbital position variation, as expressed by

$$\delta r = \epsilon p_V$$

where ϵ is some small parameter. Marec derives and makes good use of the following vectorial equation for the primer vector (Ref. 2, p. 111),

$$p_V = P_{\epsilon_1}, v + \frac{h\times P_e}{\mu} + \left[\frac{P_e\times v}{\mu} + p_h\right]\times r - p_M\frac{2r}{na^2} \tag{A1}$$

This is related to Eq. (11) for δr by the following relations:

$$\epsilon p_M = -\left(\frac{na^2}{2}\right)\frac{\delta a}{a}$$

$$\epsilon p_{\epsilon_l} = \frac{\delta M}{n} + \frac{ah}{\mu e}\hat{b}\cdot\delta e$$

$$\epsilon p_h = \frac{h\times\delta h}{h^2} \equiv \delta\theta_\perp$$

$$\epsilon P_e = \frac{\mu a}{h^2} h\times\delta e$$

It follows that

$$\frac{\epsilon}{\mu}[h\times P_e + (P_e\times v)\times r] = -a\delta e\left(\hat{e} + \frac{\mu e}{h^2} r\cdot\hat{b}\hat{b}\right)$$

$$+\delta\theta_\parallel \times r - \frac{ah}{\mu e}(\hat{b}\cdot\delta e)v$$

where $\delta\theta_\parallel = \delta\theta - \delta\theta_\perp = \hat{h}e^{-1}\hat{b}\cdot\delta e$.

Marec's Eq. (A1) has the drawback of involving constants that are not entirely independent of one another or subject to direct geometrical interpretation. There is some advantage, therefore, to using Eq. (11) instead.

References

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Optical Measurements and Attitude Motion of HERMES After Loss of Stabilization

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Introduction

THE three-axis stabilized HERMES communications satellite relied on the gyroscopic stiffness of a momentum wheel for roll and yaw stability. The satellite and its control system are described in Ref. 1, and its configuration is shown in Fig. 1. The basis for stability is essentially the dual-spin principle with a minimum moment of inertia configuration, the central body and solar sails being the "platform" and the momentum wheel the "rotor." If the wheel despins, a loss of attitude stability results.

On day 329 of 1979, an Earth sensor malfunctioned and, in conjunction with degraded telemetry and batteries the wheel despun, resulting in loss of attitude stability and ultimately shutdown of the satellite. The satellite also lost its rf transmit and receive capability shortly after the malfunction. From day 331 of 1979 to mid-1980, the sun's reflections from HERMES were periodically recorded at the Ground-Based Electro Optical Deep Space Surveillance (GEODSS) experimental test site (ETS) in New Mexico.[2] During this time, HERMES made a transition from the unstable state to a flat spin about the yaw axis, and its flat spin decayed to zero.

This Note describes the ETS observations, the correlation with theoretical knowledge, and the deduced attitude motion.

Sequence of the Attitude Motions

Immediately following the malfunction, the onboard controller went to a failure-protect mode which held wheel speed constant. A roll-yaw nutation cone of about 20 deg and a slow pitch rotation (3.1×10^{-3} rad/s) were then induced by thrusters. Within 15 min, the solar array was unable to track the sun, resulting in a battery drain. The batteries were degraded after nearly 4 yr of operation, and provided momentum wheel operation for only a few hours. Thereafter, the wheel lost drive power. The following phases of attitude motion then evolved. The pitch, roll, and yaw axes are the minimum, intermediate, and maximum moment of inertia axes, respectively, and the corresponding moments of inertia are 72, 833, and 858 slug-ft^2.

Despin of the Momentum Wheel

After loss of power, the wheel despun within about 30 min. Its stored momentum (about 16.8 ft-lb-s) transferred to the satellite body, causing a substantial rotation about pitch (about 0.238 rad/s). Numerical simulation indicates that the 20-deg nutation stayed constant during wheel despin.

Change from Spin About Pitch to Spin About Yaw

Energy sink theory applies for this phase. The dynamic state at the outset (spin about pitch with a 20-deg nutation) is near the maximum energy state. The minimum energy state is pure (flat) spin about yaw. The dissipative forces which induce the satellite toward its minimum energy state are

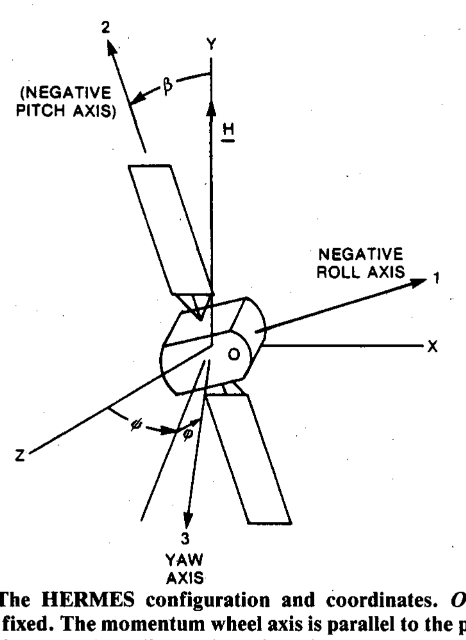

Fig. 1 The HERMES configuration and coordinates. *OXYZ* are inertially fixed. The momentum wheel axis is parallel to the pitch axis. Each of the two solar sails consists of a thin kapton blanket covered with solar cells, and a supporting silver-plated boom.

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